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CHRISTOPHORVS CLAVIVS

THE MATHEMATICS TEACHER

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The Efficiency of Decomposition Compared with That of Equal Additions as a Technique in Subtraction of Whole Numbers

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The Question at Issue

IN ORDER TO UNDERSTAND fully the import of this experiment, it is necessary that the reader have clearly in mind the different methods of subtraction.

First, there are, in simple subtraction, two methods, depending upon whether we say "from" or "and," called respectively subtractive and additive and illustrated as follows:

In the example $\begin{array}{c}9\\ \underline{5}\end{array}$ if we say, "5 from 9 (9 less 5 or 9 minus 5)" we are employing the subtractive method. But if we say, "5 and ? are 9 or ? and 5 are 9" we are employing the additive method.

Now when we come to compound subtraction with larger numbers where borrowing, so-called, is usually employed, we are confronted with two other methods depending on whether we decrease the second-place number in the minuend or increase the second-place number in the subtrahend.

Then, when the two methods of simple subtraction, the additive and subtractive, are used in connection with the two methods just

mentioned in compound subtraction, we really get four methods.
The four methods with their various names are:

When second-place minuend figure is decreased by 1.
1. Subtractive-Borrowing, Decomposition, First Italian.
2. Additive-Borrowing.*

When second-place subtrahend figure is increased by 1.
3. Subtractive-Carrying, Equal additions, Second Italian.
4. Additive-Carrying, Austrian.

The above four methods are illustrated from the following example:

71	Minuend Figure decreased	
<u>59</u>	Decomposion	(*)
	9 from 11 = ?	9 and ? are 11.
	5 from 6 = ?	5 and ? are 6.
	Subtrahend Figure Increased	
	Equal Additions	Austrian
	9 from 11 = ?	9 and ? are 11.
	6 from 7 = ?	6 and ? are 7.

In an effort to find out whether all persons used either the additive or the subtractive method in simple subtraction a number of tests on the 100 simple subtraction facts were given to high school graduates and University graduates (over 2000 tests in all). Each student was asked to state whether he used the subtractive or the additive method or both. (The methods had been previously explained to them.)

This is what was found:

1. Some said they used the subtractive throughout.
2. Some said they used the additive throughout.
3. Others said they used both methods.

* This method, as far as the knowledge of the writer goes, is not taught by any teacher, hence has no trade name. This is the reason why so many writers mention only three methods. See J. C. Stone in "How We Subtract" and W. G. Osburn in "How Shall We Subtract"? in Jr. of Ed. Research, 1927. That this method is used by students is revealed by tests. On the basis of use then there are four methods in this country. The writer is of the opinion that this method is adopted by children who have been taught the Austrian (additive-carrying) and later from mingling with other groups brought up on the decomposition method (subtractive-borrowing) learn the borrowing part of that method but still cling to their additive language. This has sometimes been called the Austrian method also, but should not because it does not involve carrying the 1 to the subtrahend figures.

The latter group was the largest in number. Upon further questioning it was found that they used the subtractive method on the easier subtraction facts such as $\begin{array}{r}6\\ \underline{1}\end{array}$ and $\begin{array}{r}5\\ \underline{2}\end{array}$ but the additive method on the more difficult ones such as $\begin{array}{r}15\\ \underline{8}\end{array}$, $\begin{array}{r}17\\ \underline{9}\end{array}$, and those of which they were not certain.

This, no doubt, comes from the fact that many teachers begin with the additive language from addition when first teaching subtraction as follows:

When first teaching the subtraction fact $\begin{array}{r}5\\ \underline{2}\end{array}$, they begin by asking, "2 and ? make 5" in this form, $\begin{array}{r}2\\ \underline{?}\\ 5\end{array}$; then," 2 and what makes 5" in this form, $\begin{array}{r}5\\ \underline{2}\\ ?\end{array}$. This is the regular additive subtraction. After that, some teachers use this additive subtraction all the way through in their subtractions; whereas others adopt the subtractive language later and say, "2 from 5 to the form, $\begin{array}{r}5\\ \underline{2}\end{array}$. This may account for the fact that so many students use the subtractive method on the easier subtraction facts and the additive on the more difficult ones (recalling the first stages of their learning subtraction).

At any rate there is not a distinct cleavage between students on the basis of whether they use additive or subtractive subtraction because the same student uses both very often. Hence we cannot very well make an issue out of the question whether we should use additive or subtractive subtraction, and I do not think that it matters very much which is used.

The truth of the matter is that perhaps we all subconsciously use a little of both methods depending upon whether the remainder is small or large compared with the subtrahend. That is to say when we subtract 8 from 9, we may say, "8 from 9" but subconsciously think, "8 lacks but 1 of being 9" or "8 and 1 are 9, therefore the difference is 1": whereas, on the other hand, if the question is 1 from 9 we can easily go down one step from 9 and

think, "1 from 9" rather than think, "what added to 1 makes 9?"

However this may be, the question at issue in subtraction is not between additive and subtractive subtraction but in compound subtraction between whether we should borrow (so-called) from the numbers in the minuend or increase the numbers in the subtrahend. Both of these methods are rarely if ever, used by the same student.

For the sake of brevity, in this experiment, these two methods shall be called respectively, decomposition and equal additions.

The object of this experiment is, then, to find out if there is any intrinsic difference in efficiency between these two methods of compound subtraction.

Method of Procedure

Several experiments in the past have been carried out in an endeavor to ascertain which of the above methods was the better. None of these, however, have been free from the influence of such factors as differences in intelligence, differences in subtracting ability, and differences in previous practice. In none of these has the ability at decomposition or the ability at equal additions been isolated from the many other abilities necessarily tied up with them.

The writer has attempted to segregate the measurement of these abilities by constructing two tests: one consists of the 100 subtraction facts, the other is made up of the same subtraction facts but put together in 30 examples so as to involve decomposition or equal additions. (See following pages for the tests.)

It is clear upon examination of an example such as $\begin{array}{r}12043\\ \underline{8765}\end{array}$ that the two main abilities involved are the abilities to subtract the facts, $\begin{array}{r}13\\ \underline{5}\end{array}\ \begin{array}{r}13\\ \underline{6}\end{array}\ \begin{array}{r}9\\ \underline{7}\end{array}\ \begin{array}{r}11\\ \underline{8}\end{array}$. (If equal additions method is used it would be $\begin{array}{r}13\\ \underline{5}\end{array}\ \begin{array}{r}14\\ \underline{7}\end{array}\ \begin{array}{r}10\\ \underline{8}\end{array}\ \begin{array}{r}12\\ \underline{9}\end{array}$) and the abilities at decomposition or equal additions, depending on the method used.

Both of these tests were given to 22 groups, 4 groups of university graduates and former teachers totalling 89 students, 6 groups of Normal College students who were high school graduates totalling 140, and 12 groups of students from the 5th, 6th and 8th

grades of various Chicago schools totalling 464 students. In all 1386 tests were given to 693 students. Of these 355 used the equal additions technique* and 338 the decomposition technique. The test on the 100 subtraction facts was given first in every case. This was followed by the test on the 30 examples during the same period in nearly all of the groups. In some of the Normal College groups the two tests were given on consecutive days or in the same week. The variable time method was used so that each student taking the test did all of the examples of the test, the time being recorded as he finished them.

As each student took both tests we have but to compare his results in the 100 facts with his results in the 30 examples to get a measure of the relative efficiency of the two methods of subtraction. That is to say, if we subtract the time used on the 100 facts from the time used on the 30 examples we have the time used at decomposition or equal additions as all the other factors such as intelligence, previous practice, and ability in the facts of subtraction were present in both of the tests.

On the side of accuracy the same condition holds. If the decreased % of accuracy on the examples is subtracted from the % of accuracy on the 100 facts, the difference is due to the presence of decomposition or equal additions alone, the other factors again being present in both tests.

Thus the extra time used on account of decomposition or equal additions is isolated and measured. Also the number of errors made due to the decomposition and equal additions can be singled out and measured.

The abbreviations, decomp. and eq. add. will be used hereafter in the tables. The tests and tables of results follow.

THE 100 SUBTRACTION FACTS IN ORDER OF DIFFICULTY

4	8	0	5	7	2	1	3	4	6	8	9	5	8	4	3	1	10	3	7
4	8	0	5	0	2	1	1	2	6	7	9	4	0	3	0	0	5	3	7
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8	4	7	4	6	5	12	9	2	3	6	5	7	2	9	5	10	6	6	8
4	1	4	0	3	1	6	0	1	2	0	0	6	0	8	3	3	5	4	1
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
10	10	7	12	9	10	16	6	8	8	6	8	9	5	10	7	10	9	14	10
9	1	1	8	5	7	8	2	5	3	1	6	1	2	3	5	4	2	7	2
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—

* In this experiment this means the method where the subtrahend figures are increased.

9	7	7	8	9	11	18	11	11	10	12	11	14	9	13	11	13	15	15	13
7	3	2	2	4	6	9	5	7	6	7	8	5	6	6	4	8	7	9	4
12	13	17	15	12	13	17	9	12	13	12	14	15	11	14	14	16	16	11	11
9	7	8	6	4	5	9	3	5	9	3	8	8	2	9	6	7	9	3	9

THE 30 EXAMPLES INVOLVING DECOMPOSITION OR EQUAL ADDITIONS

136	110	1071	726	200	13021	6002	17736	118905	110200
44	15	257	388	19	6002	2013	8759	39226	40312
137	110	1162	812	500	12843	8005	18637	151903	120300
75	91	437	474	37	6005	1038	9868	68568	72351
156	130	1134	631	600	12693	8004	17656	142405	130200
62	64	815	585	46	8009	3069	9698	98147	43936

TABLE I.—ACCURACY RESULTS ON THE 100 SUBTRACTION FACTS

Scores	*Frequencies* Decomp.	*Eq. Add.*
100	196	196
99	73	95
98	37	38
97	13	10
96	8	3
95	3	2
94	1	3
93	3	
92		1
91		4
90		
89		
88	1	1
87		1
86		
85		1
84	1	
83	1	
73	1	
	N=338	355
	Mean= 99.03%	99.08%
	P.E.= ±1.41%	±1.2%

TABLE II.—ACCURACY RESULTS ON THE 30 EXAMPLES

Scores	*Frequencies* Decomp.	*Eq. Add.*
30	38	60
29	50	73
28	61	65
27	44	41
26	29	35
25	28	24
24	15	9
23	22	13
22	10	9
21	6	3
20	8	7
19	6	3
18	2	3
17	3	
16	3	2
15	5	1
14	2	2
13	1	1
12		2
11	1	
10	1	
9		1
8	1	1
7		
6		
5	1	
4		
3		
2	1	
N	=338	355
Mean	= 85.9%	89.3%
P.E.	=±9.3%	±6.9%

TABLE III.—TIME RESULTS ON THE 100 SUBTRACTION FACTS

Time Scores	*Frequencies* Decomp.	Eq. Add.
9:00–	4	5
8:50–8:59	1	
8:40–8:49		1
8:30–8:39	1	3
8:20–8:29	2	
8:10–8:19		
8:00–8:09	2	
7:50–7:59	4	
7:40–7:49	2	1
7:30–7:39	1	
7:20–7:29	5	
7:10–7:19	4	
7:00–7:09	4	2
6:50–6:59	1	4
6:40–6:49	1	3
6:30–6:39	4	7
6:20–6:29	3	5
6:10–6:19	4	4
6:00–6:09	6	7
5:50–5:59	4	4
5:40–5:49	7	10
5:30–5:39	5	9
5:20–5:29	3	7
5:10–5:19	5	8
5:00–5:09	5	10
4:50–4:59	5	9
4:40–4:49	9	10
4:30–4:39	8	12
4:20–4:29	13	13
4:10–4:19	11	15
4:00–4:09	8	18
3:50–3:59	6	7
3:40–3:49	9	17
3:30–3:39	13	14
3:20–3:29	22	11
3:10–3:19	13	22
3:00–3:09	19	11
2:50–2:59	14	13
2:40–2:49	16	19
2:30–2:39	12	20
2:20–2:29	14	10
2:10–2:19	16	13
2:00–2:09	15	9
1:50–1:59	16	6
1:40–1:49	8	8
1:30–1:39	10	6
1:20–1:29	2	2
1:10–1:19	1	
N =	338	355
Med. =	3:21	3:43
P.E. =	±1:12	±1:02

TABLE IV.—TIME RESULTS ON THE 30 EXAMPLES

Time Scores	*Frequencies* Decomp.	Eq. Add.
19:00–19:29		1
18:30–18:59		
18:00–18:29		
17:30–17:59		2
17:00–17:29		
16:30–16:59		4
16:00–16:29	1	2
15:30–15:59		
15:00–15:29	1	
14:30–14:59	2	6
14:00–14:29	9	1
13:30–13:59	3	1
13:00–13:29	1	5
12:30–12:59	3	5
12:00–12:29	6	2
11:30–11:59	6	4
11:00–11:29	6	6
10:30–10:59	8	8
10:00–10:29	12	11
9:30– 9:59	9	9
9:00– 9:29	20	13
8:30– 8:59	12	5
8:00– 8:29	14	18
7:30– 7:59	14	16
7:00– 7:29	23	18
6:30– 6:59	20	22
6:00– 6:29	19	17
5:30– 5:59	34	35
5:00– 5:29	18	19
4:30– 4:59	22	31
4:00– 4:29	21	22
3:30– 3:59	20	28
3:00– 3:29	22	26
2:30– 2:59	7	14
2:00– 2:29	3	2
1:30– 1:59	2	2
N =	338	355
Med. =	6:32	5:58
P.E. =	±2:01	±2:12

From the results of the above tables we get the following

Summary

	Decomp.	Eq. Add.
Mean accuracy on the 100 subtraction facts	99.03 ± 1.41	99.08 ± 1.2
Mean accuracy on the 30 examples	85.9 ± 9.3	89.3 ± 6.9
Difference, or loss in accuracy, due to the technique used	13.13%	9.78%
Median time on the 100 subtraction facts	$3{:}21 \pm 1{:}12$	$3{:}43 \pm 1{:}02$
Median time on the 30 examples	$6{:}32 \pm 2{:}01$	$5{:}58 \pm 2{:}12$
Difference, or loss in time due to the technique used	3:11	2:15

The large P.E.'s in the 30 examples above are due to the fact that four 5th grade groups were included among whom there were many very slow and inaccurate students causing a very large deviation in many of the scores.

Leaving out these four groups which made up about one half of the population, the averages and P.E.'s are as follows:

	Decomp. (210 cases)	Eq. Add. (141 cases)
Mean accuracy on the 100 subtraction facts	99.3 ± 0.74	99.5 ± 0.55
Mean accuracy on the 30 examples	89.3 ± 6.1	92.5 ± 5.3
Difference, or loss in accuracy, due to the technique used	10.0%	7.0%
Median time on the 100 subtraction facts	$2{:}50 \pm 0{:}38$	$2{:}52 \pm 0{:}40$
Median time on the 30 examples	$5{:}26 \pm 1{:}14$	$4{:}36 \pm 1{:}14$
Difference, or loss in time due to the technique used	2:36	1:44

In both of the above summaries it is seen that the difference is in each case in favor of the equal additions method.

In accuracy it amounts to 3.35% and in time 56 seconds. In untechnical terms this means that each student using the decomposition method missed on an average of one whole example more than did each student using the equal additions method and at the same time he also consumed about a minute's more time.

To see if there was any difference in results among the various groups of students, the mean accuracies and median time scores of each group has been calculated with the following results:

In the University Graduates Groups	Decomp. (56 cases)	Eq. Add. (33 cases)
Mean accuracy on the 100 facts	99.4	99.7
Mean accuracy on the 30 examples	91.3	95.3
Difference	8.1%	4.4%
Median time on the 100 facts	2:41	2:25
Median time on the 30 examples	4:20	4:05
Difference	1:39	1:40

In the Normal College Groups	*Decomp.* (87 cases)	*Eq. Add.* (53 cases)
Mean accuracy on the 100 facts	99.3	99.6
Mean accuracy on the 30 examples	90.6	94.0
Difference	8.7%	5.6%
Median time on the 100 facts	2:21	2:31
Median time on the 30 examples	4:52	3:48
Difference	2:31	1:17
In the Public School Groups	*Decomp.* (195 cases)	*Eq. Add.* (269 cases)
Mean accuracy on the 100 facts	98.8	988.
Mean accuracy on the 30 examples	88.3	92.3
Difference	10.5%	6.5%
Median time on the 100 facts	4:31	4:18
Median time on the 30 examples	8:31	6:56
Difference	4:00	2:38

In each of the above groups the differences again are all in favor of the equal additions method except in the case of the time for the University graduates group in which case it was about the same. In this group the one variable, accuracy, showed a difference between the methods of 3.7%. In the Normal College group the accuracy difference was 3.1%, the time difference, 1′14″. In the public school group the accuracy difference was 4% and the time difference, 1′22″.

Conclusion

This experiment seems to show that, within its scope, subtraction by the equal additions method is, (due to its own specific technique) about 3.3% more accurate than subtraction by the decomposition method and at the same time about 14.3% faster as shown by the ratio 56″/6′32″,

$$\frac{\text{the time gained over the}}{\text{median time taken by the decomposition group}}$$

Whether we can add these two and say that equal additions is 17.6% more efficient than decomposition is not for this article to state. We will leave that to the expert statistician.

Other experiments on a larger scale should be undertaken to verify or disprove this conclusion.

Laboratory Work in Geometry

By R. M. McDill, *Hastings College, Hastings, Nebraska*

Two questions are raised by our subject: first, is it possible to teach geometry as a laboratory science and, if so, is it desirable? With certain limitations, my conclusion is that we may answer both of these questions in the affirmative.

I presume that none will object to the long accepted principle that geometry is the resultant of *sense-perception* and abstract thought. I wonder if in our desire to develop the power of abstract thought, we are, in our failures, forgetting the necessity of the other necessary element, sense perception. Most teachers will agree that pupils come to the high school after eight years of arithmetic lamentably ignorant of mensuration. Let me give just one illustration: I asked 101 high school graduates how to find the area of a triangle when the three sides are given. Three pupils answered. One said, "It is something about the sum of the sides." Another said, "It is something about a square root." And one gave the formula correctly. If this is the condition of our high school graduates, what may we say of the graduates from the grades?

We wish to make little philosophers who not only practice ideal reasoning, but are expert in invention so that original exercises may be mastered. With an ever increasing number of pupils with only ordinary ability, and little interest in education, is it any wonder that we sometimes become discouraged? It is significant that every one of the prize winning teacher writers in the recent Delineator contest on the subject, "What is the matter with teaching?" complained of the size of classes. It is more and more difficult to give that individual attention to pupils which a subject like geometry demands. I am not entirely ignorant of the recent attempts of certain over crowded schools to prove that teaching in large sections is as efficient as in small sections. This only reminds me of one of my friends who delights in "proving" that $2=1$.

It is more and more difficult to get the pupil to give the necessary time to his lessons. When my own son was a pupil in high school,

I found that his school day was almost entirely taken up with recitations, manual training, physical training, and other high school activities, leaving perhaps not more than one period for study. The day was gone and everything done but patient, plodding, time-taking study. If any of you still believe that the average high school boy or girl puts time on his preparation, read the North Central report of the questionnaire answered by 6826 pupils from North Central territory. It seems from this report that the average pupil spends on the average lesson somewhere between 30 and 45 minutes. Is this enough for the kind of thinking demanded by such a subject as geometry?

In laboratory geometry, the pupil works (for the time being, at least) as an individual. He works with his hands, at a table, using square, protractor, compass, rule and scissors. An experiment is a question put to nature. The scientific spirit demands that we be on the alert to put questions to nature and that we be honest in recording the results we really get, not necessarily what we think we ought to get.

Laboratory geometry involves individual work, experimental work and the construction of drawings; and while these elements overlap, it is possible to emphasize any one of these three—it may determine everything more or less accurately by measurement, it may become more and more, simply individual work, or it may develop into mechanical drawing. It is possible to carry any one of these to an extreme.

Before going further, we might notice the general fear that such work will lower the standards of mathematical instruction. In the October 1926 MATHEMATICS TEACHER, D. E. Smith says that the chief purpose of demonstrative geometry is to show the application of logic to mathematical statements; if demonstrative geometry is not considered largely by itself, it will remain, if at all, as a feeble memory of the world's effort to show how truth is logically established—in the mathematical sciences. Poincare once said that, "If geometry were an experimental science, it would not be an exact science. It would be subject to continual revision." De Morgan stated that "there never has been, and until we see it, we shall never believe that there can be, a system of geometry worthy the name which has any material departures from the plan laid down by Euclid." Lardner warns that, "Euclid once superseded, every teacher would esteem his own work the

best, and every school would have its class book. All the rigor and exactitude which have so long excited the admiration of men of science would be at an end." (Quoted by P. S. H. in Christian Science Monitor, Sept. 20, 1926.)

These quotations should warn us that if experimental work is to be encouraged, it must be to supply that which is essential in sense perception, and as an aid, not as a substitute, for abstract thought.

Geometry is, I believe the most difficult study the average pupil will pursue. It demands an ability to read not demanded by any other subject, careful discrimination between that which is assumed, and that which is to be proved, specific, definite, general statements and true results; it also, if rightly studied, demands an amount of invention not required by most studies. It cannot be learned in a parrot like way. It is these things which make it difficult, and it is the same elements which make it of value. I am inclined to think that our geometries have attempted entirely too much, especially in difficult original exercises. Many times the sentences are too long and involved, the author attempts to say too much in one sentence or one theorem.

Ask your pupils to prove that all right angles are equal or that a straight line can intersect a circumference in only two points, or some similar theorem. If you demand that the pupils lead, they are bewildered, while if you lead, they sit back comfortably in their seats and let you prove what is of no interest to them. The conclusion is simpler than the method of proof.

The average text book, in arguing that principles should be reasoned out rather than taken by intuition, gives one or two optical illusions showing that mistakes may be made in intuition, but never hints that any one could make a mistake in formal reasoning. Let any high school teacher go through an average pile of examination papers, and I believe he will agree that as many terrible blunders are made in the attempt at formal reasoning as in experimental results or results obtained by intuition.

Teachers of chemistry or physics or manual training demand that they get their pupils for two or three *consecutive* periods. That is something I always wanted to try with a geometry class whether it be taught by a laboratory plan or not. But in the more than twenty years I taught geometry, I never had this opportunity. Schedule making is difficult enough, and our executives

do not want any added trouble, but perhaps in the case of beginning geometry we should be just as persistent as are the science teachers in not only wishing for, but demanding consecutive periods for work.

Geometry is the most beautiful subject in the curriculum, yet in all my teaching I have found nothing so difficult to teach as beginning geometry. As a pupil I had liked the subject and had believed it highly valuable. But the work of many of my pupils was entirely unsatisfactory. Now don't think I am belittling my work too much. In the twenty years I have taught in Normal and College work in Nebraska, I have come in contact with the pupils from hundreds of schools, and I do not believe my pupils were worse than the rest, but they were not satisfactory. The English, the spirit and the thought were entirely above the heads of the pupils in general, and when the light began to break through, there was not time to drill in class on all that should have been accomplished. Neither employers nor pupils would stand for wholesale failures and yet the work was not what it should be.

About this time I was re-reading Heath's Mathematical Monograph, Number I; in this we are given a hint as to the probable order in which the ancients noticed the elementary facts of geometry. I also noticed that those pupils who had had a short course in constructive drawing in another department of the school seemed to be more in the spirit of the work. I had long considered taking a few of the fundamental theorems of geometry as experimental truths, and now I decided that it would be impossible to make things worse and might help if the work were founded on an experimental inductive introduction.

I was somewhat fortified in my belief by the following conditions. The word geometry means earth-measure. Geometry was originally surveying, or rather the principles back of surveying. Geometry should begin as well as end with measurement. It is possible to make a more natural introduction to the subject. It is possible to present the subject and make it nothing but arithmetic, or it is possible to present geometry in the highly polished condition in which it exists, but the writer felt that the time has come when we must make an attempt to make the work appeal more to the average student, even if it be at the expense of some very carefully made distinctions which may appeal to the mathematician.

In answer to the question, "Of what does mathematics treat?",

the elder Pascal (according to Cajori) said, that "It was the method of making figures with exactness, and of finding out what proportions they relatively had to one another."

D. E. Smith quotes with approval Hoüel as follows, "We ought at first to multiply axioms and to employ in place of demonstrations, experimental truths, analogy, and induction. . . . . The first teaching should be purely experimental, and little by little the pupil should come to see that all truths need not necessarily be derived from experience."

J. W. A. Young says "Work with a large body of axioms." "Let the standard for rigor of proof be the pupil's capacity to appreciate rigor and not the strictest rigor thus far attained."

Therefore, I felt that conditions could not be made worse and might be improved by following the above advice; so I determined that as an introduction to geometry and not as a substitute for it, to have my pupils work for a few weeks with ruler, compass, protractor, scissors, and the like. The pupils were led to take as experimental, inductive truths, the following theorems:*

1. The base angles of an isosceles triangle are equal.
2. If two straight lines intersect the vertical angles are equal.
3. Two triangles are equal if their bases and base angles are equal.
4. The corresponding sides of similar triangles are proportional.
5. The method of the last theorem was used to get the height of a telephone pole, and the story of Thales getting the height of a pyramid was told.
6. A circle is bisected by any diameter.
7. An angle inscribed in a semicircle is a right angle.
8. The sum of the angles of any triangle is 180 degrees.
9. (Construction). Draw a triangle when the three sides are given.
10. The theorems concerning congruent triangles.
11. If two adjacent angles have their exterior sides in a straight line they are supplementary.
12. Converse of last.
13. The complements of the same angle are equal.
14. The supplements of the same angle are equal.

* (Allow me to divert here long enough to say that careful questioning of good students in college leads me to believe that at least three fourths of high school pupils get absolutely no idea of *proof* in the superposition theorems of beginning geometry. The argument might just as well run: lay a geometry on an algebra, therefore they coincide, therefore a geometry is an algebra. Unless the work is in some way made concrete, such theorems are not grasped by the average pupil. While I realize that some will be startled by such a radical departure, I believe it is better to get something out of carefully made measurement than nothing out of general abstract demonstration.)

15. By sliding your square along a straight line, see how many perpendiculars can be drawn from a given point to a given line.
16. What is the shortest line that can be drawn from a given point to a given line?
17. Two lines perpendicular to the same line are parallel.
18. A line perpendicular to one of two parallel lines is perpendicular to the other.
19. If two parallel lines are cut by a transversal, the alternate interior angles are equal.
20. If two lines are cut by a transversal making the alternate interior angles equal, the lines are parallel.

By this time the pupil has enough tools (theorems) to enable him to begin deductive reasoning. A little special care on the part of the teacher is required when the change from inductive experimental results to deductive general results is made, but it can be made and made successfully. The method of making constructions should be kept ahead of the need to use the constructions. Such "supposed" constructions as are made in some well known text books are very confusing to beginners.

The pupil is now told that the principles determined by measurement should be few, simple in thought, and of wide application, and that it is desirable to use as few tools as possible. The ancients limited themselves to a straight edge and a compass.

After proving about 30 theorems by the orthodox method the pupil is allowed to take up any standard text on geometry, but is not required to re-demonstrate the theorems he has had.

The method I have outlined here has been used in scores of classes and is, I believe, a help but is surely not a cure-all. It can easily be argued that such work ought not to be necessary if our pupils are thoughtful, if they have been taught mensuration, and if they can read. But in how many cases does the pupil come to us having learned to "get by" with a certain amount of following form without thought, and by saying words rather than expressing thought?

It seems to me that since the war, we have had a revolution rather than an evolution in high school texts. A few months ago the highest paid high school teacher that I know said to me, "We have no new, good high school texts in mathematics." No doubt the amount of valuable supplementary material has increased as never before, but mixed with this is so much that is trivial.

Let me explain what I mean. A recent text tells us that the "to yield" income of a bond is counted on par (face) value. This

statement is repeated and emphasized. Another text tells us that the U. S. wheat production in 1922 was 3000 million bushels. Another text represents a group of 500,000 men by the picture of a man, the picture being seven centimeters high, and in the same connection a group of 250,000 men by a picture three and one-half centimeters high. It never seems to have occured to the author that a man is a three dimensional being and that the volumes of similar solids are to each other as the cubes of like dimensions. These texts are published by leading publishers and are exalted as being pedagogically correct. Such crudities were not found in Milne's High School Algebra, Sander's Geometry, or Wentworth's Trigonometry (second revision). Mathematics should give us a sense of quantity but the text book writer who tells us that in a recent year checks to the amount of 95 million dollars were written in the U. S. certainly does not watch the bank clearings.

We can only wonder if the new text books are going to be able to preserve the polish and logic and invention of the better texts of thirty years ago and yet be adaptable to the class of pupils we are receiving.

Going back to the question of experimental work, what I would like to see would be some such course as I have outlined used in the junior high school and then after a year or year and a half, have the pupil take up deductive work. A short time ago I was pleased and surprised to learn that the 7th grade boys in my own city are being given an elementary course in mechanical drawing. Such pupils should come to high school with a sense perception which should be a great help in geometry. The only thing I cannot see is why the girls should be slighted.

The great French philosopher and outstanding mathematician Poincare, says somewhere "The principal aim of mathematical teaching is to develop certain faculties of the mind, and among them intuition is not the least precious."

Allow me to quote Branford, "The best path to be followed by the pupil is clearly through the experimental and intuitional stages successively onwards to the scientific stage. What the race has never been able to accomplish, it is unsound to expect of the child. Only by the path of very gradual creation from the first and second species of evidence, wherein sense perception relatively decreases in quantity, while conception and abstraction gradually replace it, is it possible for either race or individual to

approach the summit of knowledge—rigorous scientific truth."

In conclusion, let me say that considering the class of pupils received by the average teacher of geometry, a few weeks of laboratory work, rather than recitation, is a great help. On the other hand, I am not in sympathy with the idea that every theorem must be tested by measurement, and while I have no quarrel with mechanical drawing in its place, I should regret to see the formal demonstration of geometry replaced by an attempt to do expert drawing.

You will perhaps forgive me if I take satisfaction in the trend towards introducing geometry with the help of such work as I have outlined in my paper, also my satisfaction in reading the list of theorems recommended for experimental, intuitional treatment, as given in the "Reorganization of Mathematics." As my work goes back to 1914, it antedates by several years most of the work done along this line in America.

Roman Numerals

By Vera Sanford

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When a thing that has had a wide use is finally restricted to a narrow field, it often happens that it becomes stereotyped, and generally it ceases to develop further. This has been the case with Roman numerals. After upwards of two thousand years of active life, they have been relegated to a few uses, and we, familiar with the conventional rules under which they now operate, are astonished when we realize that they have ever differed from their present form.

It is the purpose of this article to discuss the reasons for the inclusion of Roman numerals in the curriculum in arithmetic, and then to indicate some of the more important points in their history—not, it must be understood, with the idea that this material could or should be introduced into the classroom in the elementary school, but with the hope that it may add to the teacher's interest and explain some of the apparent contradictions in the system as we now know it. Perhaps it may stimulate the reader to look further into the matter by consulting the second volume of Professor David Eugene Smith's *History of Mathematics* or the first volume of the late Professor Cajori's *History of Mathematical Notations*.

Present Day Uses of Roman Numerals

The presence of Roman numerals in the curriculum of the elementary school is sometimes justified by their use in telling time, but this seems untenable for while there is evidence that some children begin to tell time by identifying the form of the numerals, this stage is soon passed and children are as quick to read a clock marked with the name of the firm it advertises as one marked with either Roman or Hindu-Arabic numerals.

A better reason for the inclusion of Roman numerals in the curriculum is the need of two sets of numbers to avoid confusion.

This is illustrated by the use of Hindu-Arabic numerals to mark the pages of the main body of a book while Roman ones indicate the number of the volume, of the chapter, or of the pages of the introduction and other front matter. In Italy, Hindu-Arabic numerals are used to indicate the Christian era while Roman ones stand for the Fascist year. Postmarks in France and perhaps elsewhere give the day of the month and the number of the year in Hindu-Arabic numerals while the number of the month is in Roman figures.

The use of Roman numerals in inscriptions, on the other hand, is traditional and is perhaps due to artistic considerations or perhaps to the exigencies of the stone cutting. Here, we should note that in themselves, Roman numerals are no more to be preferred than are other systems of the ancient world. In the Colleoni Chapel in Bergamo for example, a fifteenth century Italian sculptor used Herodianic Greek numerals in one of his inscriptions and this display of classical erudition has a very pleasing effect.[1] It was only because Roman numerals were in popular use at the time when Hindu-Arabic numerals were introduced into Europe that they were the ones to survive as the second system.

The Persistence of Roman Numerals

The question of the persistence of Roman numerals in Europe after the introduction of the Hindu-Arabic ones, has been accounted for by several hypotheses: the inertia that preserves familiar things and that distrusts innovations, the fact that the newer numerals were far from standardized and so could be used by an unscrupulous merchant to cheat his patrons, and the fact that in the eyes of some they were associated with the work of soothsayers and charlatans and thus had no standing in the mind of honest men.

In this connection, it is interesting to consider the case, probably a unique one, of a storekeeper in Pisa who, in today's era of "fixed prices" marks his goods with Roman numerals. When questioned, he says, "People from all over the world can read them, whatever their own numerals,—Turks, Italians, French, and even

[1] For Herodianic numerals, see Smith, *History of Mathematics*, II, 49, and the writer's *Short History of Mathematics*, 82–83. It should be noted that these numerals are particularly well adapted for inscriptions cut in stone as no curved lines are needed.

Americans." To an American who had puzzled over the European variants of 1, 5 and 7 on hotel bills and elsewhere, the scheme seemed very practical, and it called to mind the regulation of 1299 in the neighboring city of Florence that merchants should not employ the new ciphers (Hindu-Arabic numerals) but should use "clear letters" instead.

Of the reasons for the persistence of Roman numerals, it is probable that inertia was the important factor. At any rate, we have evidence in household accounts and in government records that these numerals were in active use in England in the first part of the seventeenth century.[2] In France they are said to have been used in government records a hundred years later.

As a matter of fact, if one used a counting board for computation, number symbols were important only to write the result, and the I, X, C, and M of the numerals corresponded to the lines of the counting board while the V, L, and D stood for the spaces. Thus a result would be transcribed by writing as many I's as there were counters on the unit's line, a V if a counter stood in the next space, as many X's as there were counters on the ten's line, and so forth.[3]

Further, since the ordinary computations of addition and subtraction are more simply mastered by the help of counters than by the memorizing of addition and subtraction combinations, it is not surprising that the new numbers were slow in coming into general use in the northern parts of Europe. In Italy which was in advance of the other countries in her business methods for a long period, the Hindu-Arabic numerals were in common use long before this was the case in Germany and France.

The Form of Roman Numerals

Of the various theories regarding the origin of Roman numerals, one of the most plausible is that they were derived from numerals used by the Etruscans who were the predecessors and then for a time the rivals of the Romans. Cantor suggests that the chance resemblance of the Etruscan characters to letters in the Roman alphabet caused them to shift to the Roman forms. Thus the use of C for 100 and of M for 1000 resulted from the accident that the old

[2] Sotheran's *Price Current of Literature*, 1930 shows a facsimile of accounts in Roman numerals from the Ordnance Office. The date is 1627.

[3] For a description of counter reckoning, see Smith, II, 181–192 and Sanford, *Short History*, 87–93.

symbols for these numerals were somewhat like the initial letters of the Latin words *centum* and *mille,* but were not in the first instance derived from them.

Among the symbols for 1000 were ∞, ↀ, and (|). Priscian, a writer of the sixth century, suggests that the third of these was a vertical line enclosed in parentheses, and from it, he said, were derived |) for 500, ((|)) for 10,000, and |)) for 5000. This suggests that the D for 500, which by the way was sometimes inverted as ᗡ, was derived from closing the parenthesis on the vertical line. In some cases, the enclosing arcs were shorter than was the vertical line, and small letters and capitals were used at will. Thus a book printed in London bears the date c|ↄ|ↄ clix.[4] The symbol C showed little variation, but like the D, was sometimes inverted to make a more symmetric appearance. The symbol for 50 has had several variants also. It appears as a vertical stroke rising from a small v as ↓, or from an arc as ⊥, or from a straight line ⊥.[5] In the middle ages, the symbol for 1 when used at the end of a number was frequently written as *j* instead of *i*. Professor Cajori suggests that this was to prevent raising the number by the addition of another letter.

When we bear in mind that Roman numerals have had an active existence as long as that of the Hindu-Arabic numerals, these changes are not so strange as they at first appear. Perhaps it is stranger that the changes have not been more marked.

Writing Numbers with Roman Numerals

The subtractive principle that is characteristic of Roman numerals seems to have been borrowed from the Etruscans. Its origin has been suggested by the idea that it is easier to count backward by one than to count forward by four. Thus it is easier to think of nineteen as one less than twenty (cf. the Latin *undeviginti*) than to think of it as four more than fifteen. The number four itself is a simpler case, but whether it was for this reason or whether the Romans avoided the symbol IV because of its association with the name of Jupiter (IVPITER), this symbol did not come into common use until the fifteenth century or later.

Even after the invention of printing, people exercised their fancy in their use or neglect of the subtractive idea or indeed in

[4] Isaac Barrow, *Euclidis Elementorum*, London, 1659.

[5] Other variants are illustrated in Cajori, I, 30–31, and in Smith, II, 55–58.

their use or neglect of the symbols V and L. Among the many instances that might be quoted are the following:

Ps. XXXXXV	(Psalm 55) in Fra Angelico's picture of the Flight into Egypt. Here the customary LV would have made the line enclosing his text and its reference too short.
MCCCCIIC	from the Foundling Hospital in Florence. The numerals occur in an inscription concerning one of the founders.
M.D.XXXXIIII	from the title page of *Liber de geometria practica* by Orontius Fineus.
M.DC.IL	from the title page of the third edition of Galileo's work on proportional compasses.
MDCVC	from a medal struck in honor of Gian Domenico Cassini.

The use of a period to separate the parts of a number persisted for several centuries, and in this country it was used by a number of printing houses on title pages, books from Benjamin Franklin's press being among them.

Capelli in his dictionary of Latin and Italian abbreviations, shows that in French manuscripts of the middle ages, multiplication by twenty is indicated by a sort of coefficient-exponent notation,—IIII^{xx} for 80, VI^{xx}XI for 131. Here we have evidence of the counting by twenties preserved in the French name for 80,—*quatre-vingt.* A similar instance occurs in a manuscript of 1291 in Professor Smith's collection,—Mil ii^{c} iiij^{xx} & xi, that is, one thousand, two hundred, four score and eleven.

Numerals were sometimes combined with words as is shown in the last example. Another case in which the words are found in the middle of the number as well as at the beginning is in an inscription cut in the floor of the Sainte Chapelle in Paris:

l an de grace Mil c·c·c·c· ·quatre vints et iiii

(the year of grace 1484).

In the transition period, Roman and Hindu-Arabic numerals were sometimes used in the same numbers. A set of family records in the British Museum contains the following dates: Mij^{c}.lviii, Mij^{c}.lxi, Mij^{c}.63, 1264, 1266.[6] A later instance of the same thing is taken from English accounts of the year of the Armada (1588): v *li.* ix *s.* 4 *d.*, that is, five pounds, nine shillings, four-pence.

If we compare the rules now followed for the formation of Roman numerals with the tables given in arithmetics published a hundred and fifty years ago, we find an interesting contrast. In the 1781

[6] See Robert Steele, *The Earliest Arithmetics in England,* London, 1922, p. xviii.

edition of the arithmetic of F. Legendre[7] we find the following alternate forms: for 80, IIIIxx and LXXX; for 90, IIIIxxX and XC; for 200, CC and IIc; for 400, CCCC or IVc; for 500, Vc or D or |); for 900, IXc or DCCCC or |)cccc.

Roman Numerals and the Clock Face

The frequent use of the symbol IIII instead of IV on the clock face has called for explanation from people unacquainted with the aversion to the symbol IV which has been mentioned elsewhere in this article. This has led to the invention of the story of the king of France who wanted a clock that should be different from all others in the world. Accordingly his jeweler inserted the four-I symbol and thus set the fashion for clock faces. This ex-post-facto explanation has no foundation, and should be discarded. It seems probable that the clock face was somewhat standardized before the shorter notation for four came into general use, but a few clocks are found that have the IV notation. Among the older examples are a clock on the west front of the cathedral at Verona and one in the nave of the church of St. Germain-des-Près in Paris. A railway poster shows this marking on the great city clock, the Grande Horloge, in Rouen, but a photograph of the clock shows the customary IIII. Among modern examples of the IV form are the clocks of the Union Terminal in Cleveland and the Union Station in Chicago. Occasionally, household clocks with this notation are to be found if the pictures in advertisements are to be trusted. In contrast to these, we should consider a clock of 1546 in the city of Brescia in northern Italy. This may be the very clock of the problem: how many times does a clock strike in a day and night if it strikes the hours from one to twenty-four? The single hand makes one complete circuit of the face in twenty-four hours, the I being in the position of our III. In spite of the somewhat crowded look of the face, the subtractive principle is never used, and instead of the shorter forms, we find the following numbers: IIII, VIIII, XIIII, XVIIII, XXIIII.

It seems to be impossible to make any general statement in regard to Roman numerals and be strictly truthful unless we say simply that we can find examples of almost any usage we can devise.

[7] *L'Arithmétique en sa Perfection.* The first edition seems to have been in the latter part of the seventeenth century. The author should not be confused with the mathematician Adrien-Marie Legendre (1752–1833).

Conic Sections Formed by Some Elements of a Plane Triangle

By Aaron Bakst

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A simple application of elementary analytical geometry (within the scope of the twelfth grade of high school, or, at most, in Freshman College Mathematics) in the process of examining the properties of geometrical figures leads to some very interesting results.

Suppose we are given a triangle, the base of which is fixed in magnitude and position, and the area also given (that is, fixed). If we construct on that base all triangles having the same area, then we shall have the following proposition:

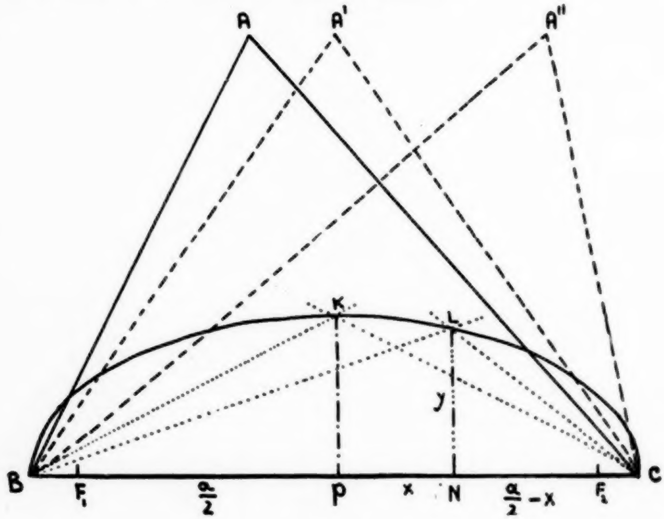

Fig. 1.

Given a triangle; its base fixed in magnitude and position. Then the locii of the points of intersection of the:

1. Medians,

2. *Perpendiculars constructed at the midpoints of the sides,*
3. *Bisectors of the angles,*
4. *Altitudes,*

of all triangles having the same area on the same base are respectively:

1. *A straight line,*
2. *A straight line,*
3. *An ellipse,*
4. *A hyperbola.*

Properties 1 and 2 are obvious. In case 1 the locus will be a straight line parallel to the base, one third the distance from the vertex opposite that base. In case 2 the locus will coincide with the perpendicular drawn from the midpoint of the base.

Let us examine property 3; that is the locus of the points of intersection of the bisectors of the angles of the triangles. Consider Fig. 1. The notation of the angles of the triangles ABC will be α, β, and γ for the vertices A, B, and C respectively. Side $BC=a$, side $AC=b$, and side $AB=c$.

$$\frac{LN}{BN} = \frac{y}{\frac{a}{2}+x} = \tan\frac{\beta}{2} \tag{1}$$

$$\frac{LN}{NC} = \frac{y}{\frac{a}{2}-x} = \tan\frac{\gamma}{2} \tag{2}$$

then multiplying (1) by (2) we have

$$\frac{y^2}{\frac{a^2}{4}-x^2} = \tan\frac{\beta}{2}\tan\frac{\gamma}{2} \tag{3}$$

which is the equation of the locus of all the points common to the bisectors BL's and LC's (bisectors of β and γ)

Transforming the equation (3) we have

$$\frac{x^2}{\frac{a^2}{4}} + \frac{y^2}{\frac{a^2}{4}\tan\beta\tan\gamma} = 1. \tag{4}$$

This is an equation of an ellipse in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The major axis of the ellipse is the base of the triangle, and the minor axis is equal to twice the length of the radius of the circle inscribed in the isosceles triangle A′BC, or

$$\frac{a}{2}\sqrt{\tan\frac{\beta}{2}\tan\frac{\gamma}{2}}\,.$$

If we construct such ellipses on the three sides of the triangle, that is if we fix the positions of the vertices of the triangle, and then take each side at a time, and on that side construct all the triangles having the same area, we obtain some very interesting results, the properties of which will be discussed in one of the following issues of the Mathematics Teacher.

The fourth property is derived from Fig. 2. We have:

$$\Delta BNC \sim \Delta KPC$$

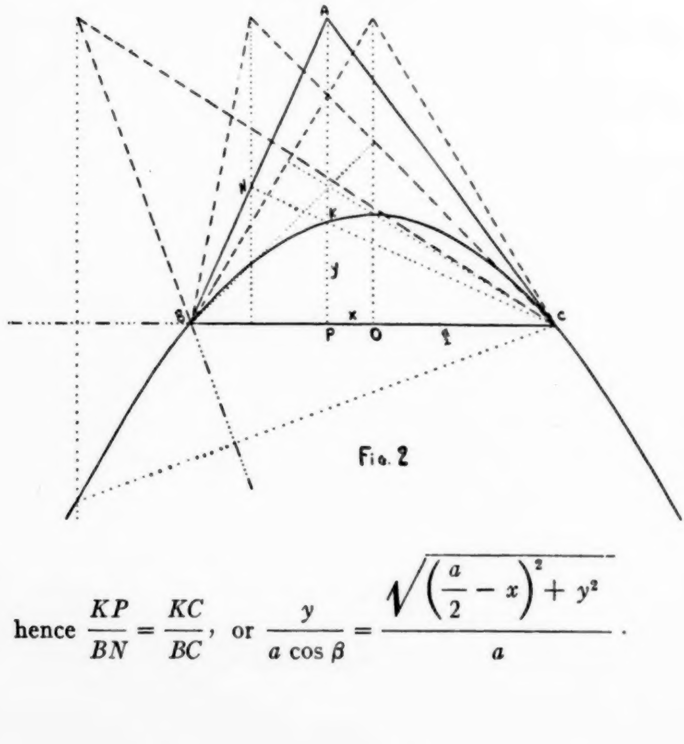

Fig. 2

$$\text{hence } \frac{KP}{BN} = \frac{KC}{BC}, \text{ or } \frac{y}{a\cos\beta} = \frac{\sqrt{\left(\frac{a}{2} - x\right)^2 + y^2}}{a}\,.$$

Square both sides of this equation, and we have

$$\frac{y^2}{\cos^2\beta} = \frac{a^2}{4} - ax + x^2 + y^2$$

or

$$x^2 + y^2\left(1 - \frac{1}{\cos^2\beta}\right) - ax + \frac{a^2}{4} = 0$$

$$x^2 + y^2\left(\frac{\cos^2\beta - 1}{\cos^2\beta}\right) - ax + \frac{a^2}{4} = 0$$

$$x^2 - y^2\tan^2\beta - ax + \frac{a^2}{4} = 0.$$

The discriminant test for a hyperbola defined by the general equation

$$Ax^2 + By^2 + 2Cxy + Dx + Ey + F = 0$$

is given by the inequality $AB - C^2 < 0$.

We, then have $-\tan^2\beta < 0$ which is true for all real values of β. We shall return to the properties of the hyperbolas of the discussed set of triangles subsequently.

The Principle of Continuity

By Francis P. Hennessey
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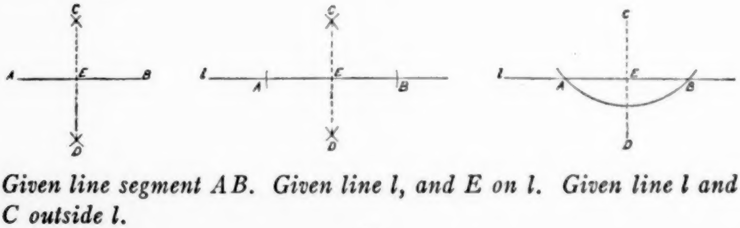
An interesting and effective manner of conducting a review in plane geometry is to make use of the Principle of Continuity whenever possible. Of the several text-books which I have had the opportunity to consult, relatively few, (D.E. Smith, Seymour, Schultze, and Nyberg) have made reference even indirectly to this important property which weaves many theorems into one. It is difficult to reconcile such an omission with the modern emphasis on functional thinking in mathematics.

Many teachers concede that the ideas discussed herein should be given a place in the presentation of their subject but are unable to do so because of the time element. It is hoped that the following development will afford for some a means of overcoming this obstacle.

The first example of continuity is found in the fundamental constructions. The two cases of constructing perpendiculars to a line, (viz. from a point outside the line, and at a point in the line) are in effect the same as the bisecting of a line segment. When we bisect a line segment we have two points given on the line, and they are the centers from which we draw the intersecting arcs. When we construct a perpendicular to a line at a point in the line, we find a ***segment*** of the line and construct the perpendicular bisector of this line segment. In constructing a perpendicular to a line from a point outside the line we draw an arc with the given point as a center and radius long enough to cut off a segment on the line. We then bisect this segment.



Given line segment AB. Given line l, and E on l. Given line l and C outside l.

A study of the three figures on page 32 will bring out the identical features present in their construction.*

Another example of the principle of continuity is found in connection with a theorem and corollary which have extensive application in the solution of certain originals. For example, *the median of a trapezoid is parallel to the bases, and equal to half the sum of the bases.*

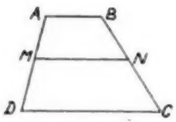


Fig. 1.

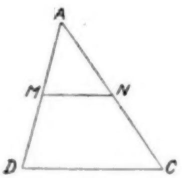


Fig. 2.

If in Fig. 1 while $\angle D$ and $\angle C$ are constant, a line moves upwards on the sides of the trapezoid to the position AB, it will decrease, and when its length becomes zero the figure will be the triangle ADC of Fig. 2. MN will take a position joining the mid-points of the sides of the triangle ADC and the original formula $MN=1/2\ (AB+DC)$ becomes $MN=1/2(0+DC)$ or $MN=1/2DC$, which expresses the theorem, *A line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.*

The second book of plane geometry contains a family of theorems on angle measurement which are readily adapted to a continuous treatment. Schultze in his "Teaching of High School Mathematics," suggests the development in the following statement:

> If we consider all arcs positive when formed by a point moving in a counter-clock-wise direction, and as negative when the point moves in a clock-wise direction, we may express all theorems relating to the measure of an angle in the one statement:—*An angle is measured by one half the algebraic sum of the intercepted arcs.*

* Even the bisection of an angle involves the construction of a perpendicular bisector of a line segment.

Below are the figures and the accompanying discussions for the seven theorems to which Schultze refers.

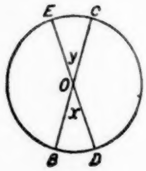


Fig. 1

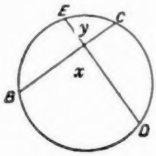


Fig. 2

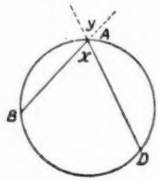


Fig. 3

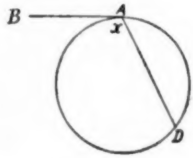


Fig. 4

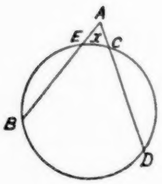


Fig. 5

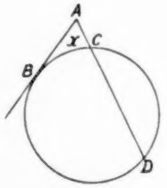


Fig. 6

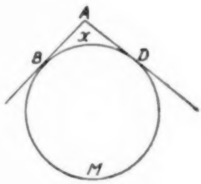


Fig. 7

Case 1. Given the circle with center O as in Fig. 1, and the two diameters BC and DE ($1\frac{1}{2}''$) intersecting at the center, forming the vertical central angles x and y.

Angles x and y are generated by the diameter BC rotating about the center of the circle in a counter-clockwise direction so that the extremities B and C move in an accepted positive direction over the arcs BD and CE respectively.

$$\begin{aligned} \angle\, x &\overset{m}{=} BD \\ \angle\, y &\overset{m}{=} CE \\ \therefore\ x + y &\overset{m}{=} BD + CE \\ 2x &\overset{m}{=} BD + CE \\ \therefore\ \angle\, x &\overset{m}{=} \tfrac{1}{2}(BD + CE). \end{aligned}$$

Case 2. Imagine for the purpose of this illustration that the two diameters in Case 1. are elastic and made fast to the circumference. Now loosen the end B, take up 1/4 of an inch and revolve the line about C until it can be fastened again to the circle as a chord, at the left of its original position. Likewise loosen D, take up 1/8 of an inch and move it to the right so that it can be fastened again as a chord. We now have Fig. 2.

The theorem represented here is usually stated as follows: *An angle formed by two chords intersecting within a circle is measured by half the sum of its arc and that of its vertical angle.*

As in Case 1. arcs BD and CE are positive. By the conventional proof it may be established that $\angle x \overset{m}{=} \frac{1}{2}(BD+CE)$.*

Case 3. Next in order is the theorem: *An inscribed angle is measured by half the intercepted arc.*

Fasten the chords BC and DE of Fig. 2 at their point of intersection. Designate this point, A, and by stretching the chords attach it to the circumference. We now have Fig. 3 in which angle BAD (or x) is an inscribed angle. The arc CE has disappeared. (become zero).

The relationship	$\angle\ x \overset{m}{=} \frac{1}{2}(BD + CE)$
becomes	$\angle\ x \overset{m}{=} \frac{1}{2}(BD + 0)$
or	$\angle\ x \overset{m}{=} \frac{1}{2}BD$

The following questions may now be asked:

1. Has the arc BD changed?
2. How has $\angle x$ changed?
3. How has $\angle y$ changed?
4. Has arc CE changed?

Case 4. If B is loosened and moved around until the chord BA becomes a tangent at A, the arc BD will change to major arc AD, and arc CE will still be zero. (see Fig. 4).

The theorem involved is: *An angle formed by a tangent and a chord is measured by half the intercepted arc.*

Here the relationship	$\angle\ x \overset{m}{=} \frac{1}{2}(BD + CE)$
becomes	$\angle\ x \overset{m}{=} \frac{1}{2}(\text{major arc } AD + 0)$
or	$\angle\ x \overset{m}{=} \frac{1}{2}(\text{major arc } AD)$.

* Such questions as the following might lend further interest to the general discussion:—

What effect has the change from Fig. 1 to Fig. 2 had

(a) upon the point of intersection of the two lines?
(b) upon the arc BD?
(c) upon the arc CE?
(d) upon the angle x?
(e) upon the angle y?

Has the sum of the arcs BD and FC changed?
Has half the sum of the arcs BD and FC changed?

Case 5. If BA is moved back to its position as a chord and A is stretched beyond the circumference, we have the case of two secants to a circle from an external point A, as in Fig. 5.

Should the line AB be moved from the initial position AB to the terminal position AD the extremity B would move along the arc BD in a counter-clockwise direction, and the point E would generate the arc EC in a clockwise direction.

Arc BD is therefore positive and arc EC negative.

Then our original proposition $\angle x \overset{m}{=} \frac{1}{2}(BD + CE)$

becomes $\angle x \overset{m}{=} \frac{1}{2}(BD + [-CE])$

or $\angle x \overset{m}{=} \frac{1}{2}(BD - EC)$.

Case 6. When we loosen B and swing it on A as a pivot until it is tangent to the circle we have the case of a secant and a tangent. Arc BD is still positive, and arc EC has become arc BC and is

negative. Now the relationship $\angle x \overset{m}{=} \frac{1}{2}(BD + CE)$

becomes $\angle x \overset{m}{=} \frac{1}{2}(BD + [-BC])$

or $\angle x \overset{m}{=} \frac{1}{2}(BD - BC)$.

It is interesting to note the change in the positions of the points B and E if the secant AB of Fig. 5 were to be moved in a clockwise direction. B and E would approach each other until they would coincide at B of Fig. 6, and then the secant would become a tangent.

Case 7. If points A and B remain fixed and ACD is swung around in a counter-clockwise direction, the points of intersection of the secant will move closer together, and when they meet the secant will have become a tangent. We then have the case of two tangents AB and AD drawn from the external point A forming major arc BD and minor arc DB.

Since angle x is formed by the line AB generating in a counter-clockwise direction about A to the position AD, the arc BMD is positive, while minor arc DB is negative.

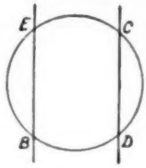
Since arc CE = minor arc DB and is negative,

$\angle x \overset{m}{=} \frac{1}{2}(BD + CE)$

becomes $\angle x \overset{m}{=} \frac{1}{2}(BMD + [-DB])$

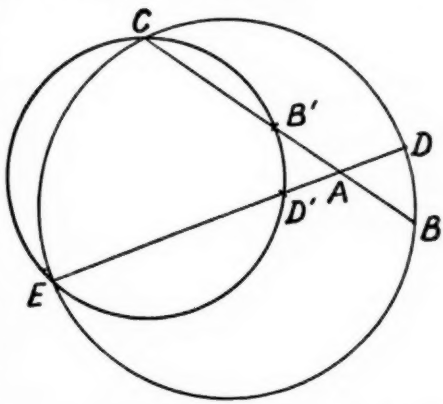
or $\angle x \overset{m}{=} \frac{1}{2}(BMD - DB)$.

Case 8. To further illustrate the continuity we may separate the two lines at A and move them around to a parallel position. The figure shows that angle x becomes zero. The arc BD is positive, arc CE is negative, and their absolute values are the same.



Therefore	$\angle x \stackrel{m}{=} \frac{1}{2}(BD + CE)$
becomes	$\angle x \stackrel{m}{=} \frac{1}{2}(BD + [-CE])$
or	$\angle x \stackrel{m}{=} \frac{1}{2}(BD - BD)$
or	$\angle x = 0$

We will now pass along to the third book and consider the following theorem: *If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.*



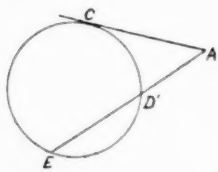
By proving triangles ACE and ADB similar and setting up the necessary proportion of the sides it is shown that $AC \times AB = AE \times AD$. On EA place D' so that $DA = AD'$, and on CA place B' so that $BA = AB'$. Connect the points $ED'B'C$. It may be shown that the opposite angles of this quadrilateral are supplementary and so a circle may be passed through these points. We now have the cas· of two secants $AB'C$ and $AD'E$ from the point A.

Proof I	Proof II
(1) Prove $\triangle ABD \sim \triangle AEC$ (2) Express the proportion $AC:AD = AE:AB$ (3) Obtain the product $AC \times AB = AE \times AD$	(1) $\triangle ABD \sim \triangle ACE$ (2) $\triangle ABD \cong \triangle AB'D'$ (3) $\therefore \triangle AB'D' \sim \triangle ACE$ (4) Form the proportion $AC:AD' = AE:AB'$ (5) Obtain by substitution $AC:AD = AE:AB$

Note that the *two* proofs contain identical elements.

The proof that the product of one of these secants and its external segments is equal to the product of the other and its external segment, is identical with the proof of the original proposition about the chords intersecting. The outline on p. 37 bears out this fact.

If AC is moved about the point A to a position of tangency, AC will become equal to AB', and we will have the case of a secant and a tangent to a circle from an external point.



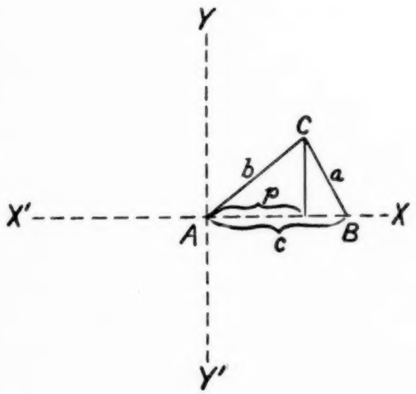
The equation $AC \times AB' = AE \times AD'$

becomes $AC \times AC = AE \times AD'$

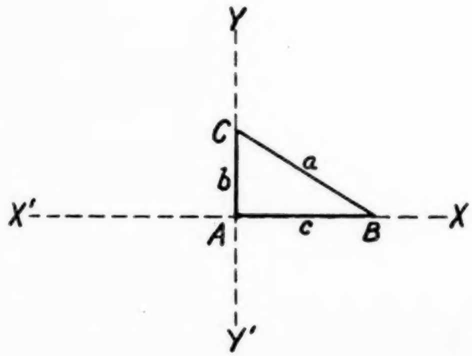
or $\overline{AC}^2 = AE \times AD'$.

Thus, *the tangent is the mean proportional between the secant and its external segment.*

A remark by Schultze in the latter part of Book 3 of his Plane Geometry, suggests the following treatment of the Pythagorean Theorem.



Since a right angle is a particular kind of an angle, the formula for the square of the side opposite a right angle, viz. $a^2 = b^2 + c^2$ is a particular case of a more general formula, for the square of the side opposite any angle. In this case triangle ABC is an acute triangle and angle A is an acute angle. The projection "p" of side b upon side c, is measured to the



right of the vertical line YY' and is therefore positive. The formula for the square of side "a" is:

$$a^2=b^2+c^2-2pc.$$

In this case the terminal ray AC coincides with YY' (see lower cut p. 38). The projection "p" of b upon c is 0, and the formula

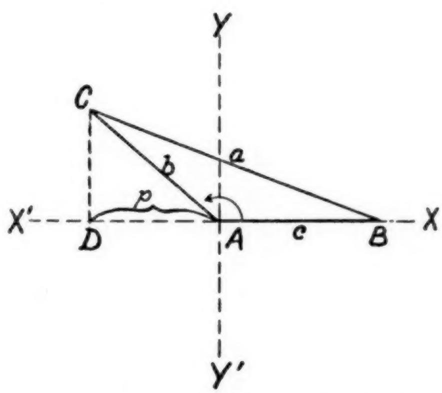
$$a^2=b^2+c^2-2pc$$

becomes by substitution

$$a^2=b^2+c^2-2\cdot 0\cdot c$$

or

$$a^2=b^2+c^2$$



In this case the terminal ray is in the second quadrant, and hence angle A is obtuse. The projection "p" of b upon c is measured to the left of YY' and is therefore negative, and the formula $a^2=b^2+c^2-2pc$ becomes by substitution

$$a^2 = b^2 + c^2 - 2(-p)c$$

or

$$a^2 = b^2 + c^2 + 2pc.$$

Therefore, the last two theorems above are special cases of the first, and we always have $a^2=b^2+c^2-2pc$.

Assuming that the class is familiar with the meaning of the cosine of an angle it may be well to show that this general formula is in fact the Law of Cosines.

In the figure $p=b \cos A$, therefore, $a^2=b^2+c^2-2pc$ becomes by substitution $a^2=b^2+c^2-2bc \cos A$.

The introductory propositions on areas in the fourth book provide further examples of continuity. The formula for the area of a trapezoid viz. $A=\frac{1}{2}h(b+b')$ may be considered as embracing that of the parallelogram and triangle.

If $b=b'$ and the angles are not right angles, then $A=\frac{1}{2}h(b+b')$ becomes $A=\frac{1}{2}h(2b)$ or $A=bh$ and the figure is a rhomboid.

If $b=b'$ and the angles are right angles, the figure is a rectangle.

If $b=b'=h$ and the angles are right angles, $A=\frac{1}{2}h(b+b')$ becomes $A=\frac{1}{2}h(2h)$ or $A=h^2$ and the figure is a square.

If $b'=0$, $A=\frac{1}{2}h(b+b')$ becomes $A=\frac{1}{2}h(b+0)$ or $A=\frac{1}{2}bh$ and the figure is a triangle.

There is nothing original in the thoughts developed in this paper. No doubt many teachers have been making profitable use of them for years. It has been my aim merely to set forth a detailed treatment in the hope that it might provide for some, material for at least an appreciation lesson.

Clavius

Christopher Clavius (1537–1612) was a German scholar who became a Jesuit and who spent the latter part of his life in Rome. His portrait gives evidence of his varied mathematical activities for the instruments hung on his wall or standing on his desk indicate his interest in astronomy and trigonometry, the drawings under his hand and the compasses he holds show his work in geometry, and the books piled before him may be looked upon as being his arithmetic, his algebra, and his commentaries on the works of earlier writers. Clavius was influential through his power as a teacher and through the popularity of his publications rather than for his discoveries in the field of mathematics. What is perhaps his greatest achievement was in connection with the adoption of our present calendar. Here, again, his rôle was to interpret and execute the ideas of others. It is unnecessary to discuss the details of the omission of ten days from the year 1582 in order to bring the calendar year into harmony with the astronomical year or the replacing of the Julian calendar sponsored by Julius Caesar by the Gregorian calendar of Pope Gregory XIII, but it is interesting to note that Clavius was summoned to Rome to explain the theory of the innovations. Ball notes* that Clavius rejected the proposal of omitting a leap year day once in 134 years and substituted the omission of three days in a 400 year period.

* *Mathematical Recreations and Essays*, London, 1892; 1919 ed., p. 445.

The Fourth Dimension:

AN ARTICLE FOR HIGH SCHOOL GEOMETRY TEACHERS

By Anice Seybold, *Urbana, Illinois*

The idea of a fourth dimension seems to have about the same kind of fascination for the lay mind as the famous old problems of duplication of the cube, trisection of the angle, and quadrature of the circle had. Certain things which would follow if we could live in four dimensional space seem quite remarkable to one who has not given the subject enough thought to realize that the actual existence of such space would make it possible to draw four mutually perpendicular lines through a given point, whereas in three dimensional space we can draw only three. These apparent absurdities of four dimensional space appear absurd only when we try to make them conform to our three dimensional experience.

We are told that in four dimensional space it would be possible to remove the contents of an egg without breaking the shell, to turn a flexible sphere inside out without stretching or tearing, and to separate the links of a chain without breaking them—all of which is perfectly true. We know that the egg, the sphere, the links are closed in three dimensions but open in the fourth, and it is through this extra dimension that these things are accomplished. We could remove the contents of an eggshell through the opening in the fourth dimension just as we could remove anything from a two dimensional enclosure through the third dimension. We could see inside a sphere just as we can see the center of a circle in a plane. We are prevented from visualizing these possibilities however by the fact that our senses limit us to a three dimensional world. Nothing that we have ever experienced has four dimensions as far as we know. It has been suggested, however, that we ourselves may have a very slight thickness in the fourth dimension which our limited sense organs are unable to detect.

The idea of a fourth dimension has stimulated imagination to such an extent that several popular treatises have been written in which imaginary spaces of one and two dimensions are peopled respectively with points and plane figures having perspective

faculties only in the dimensions of their spaces. A point removed from its one dimensional space, a line, by a superior two or three dimensional being would be invisible to his former neighbors. In exactly the same way a square moved out of its two dimensional world into the third dimension would be invisible to its former neighbors. If we could but move into the fourth dimension could we not disappear as if by magic from our companions simply by moving out of the three-space in which they were located? It is this very intriguing thought that spiritualists have seized upon to explain some of their spiritual manifestations.

There seems to be some confusion among writers as to whether a right glove becomes a left glove upon being turned wrong side out or upon being turned over in four dimensional space. H. P. Manning in the introduction to "Fourth Dimension Simply Explained" states the correct theory: "A right glove turned inside out in our space becomes a left glove, and a right glove turned over in a space of four dimensions becomes a left glove, but when the glove is turned over it is not turned inside out."

In order to see Manning's statement more clearly let us reduce the number of dimensions by one and take an example in two and three dimensional space. Let us think of two triangles lying in a plane and symmetric with respect to an axis in the plane. One triangle may be turned into its symmetric by rotation through the third dimension about the axis, or, in other words, by simply being turned over in three dimensional space. However if we wish to accomplish this without the use of the third dimension, we may turn one triangle wrong side out, so to speak, in its own plane. This feat involves the notion of points on the inside of a triangle and points on the outside of it, one set of points changing places with the other. It is necessary to realize here that the triangle is a broken line and that points on the inside of this line are not within the area inclosed by it. In order to get an idea of the results of turning a triangle inside out in the way that will give us the symmetric form (there is a way of doing it that does not give the symmetric form) let us imagine we have clipped the corner off an envelope so that the edges of the envelope and the clip form a representation of a scalene triangle. The points on the outside of the triangle lie along the outer edge of the paper. Now let us turn this corner of the envelope inside out so as to bring the points that were on the outside of the edge onto the inside of it. We

have turned the triangle inside out and the result is the symmetric form of the triangle. This illustration, of course, involves a slight thickness in the third dimension but it has the tremendous advantage of being concrete. We may now think of the same thing happening to a broken line such as a triangle which has no thickness at all in the third dimension. In this case, the triangle would be severed at a vertex, say, and one end drawn back around the outside of the triangle until it met the other end. In effect we have accomplished this: We have turned a two dimensional right handed figure into a two dimensional left handed figure, (1) by turning inside out in its own two dimensions and, (2) by rotation through an extra—the third-dimension. By analogy we find it easy to believe that a three dimensional figure such as a right glove is turned into a left glove not only, (1) by turning inside out in its own three dimensions but also (2) by rotating into an extra—the fourth-dimension.

The second way of turning a two dimensional figure inside out assumes the existence of the third dimension, and will suggest to us what happens to a glove turned inside out in the fourth dimension. It leaves the figure the same shape as before and has been illustrated by assuming a rubber band to be approximately two dimensional and twisting it on itself through 180°. This presupposes the existence of a direction not lying in the plane of the band, whereas the first method involves turning the figure back along itself in its own plane. We can now understand why turning a right glove inside out in space limited to its own three dimensions changes it into a left glove, but turning it inside out in four dimensional space (which has one more dimension than the glove has) does not change it into a left glove. This same illustration of the rubber band sheds light on the fact that a figure completely closed in three dimensions such as a flexible sphere could, in four dimensional space, be turned inside out without stretching or tearing.

Time and again teachers have explained to high school geometry classes that a point has no dimensions, but when it is moved into a first dimension it generates a line of one dimension, which, when moved into a second dimension, generates a plane of two dimensions, which, in turn, moved into a third dimension generates a solid which has three dimensions. We might extend the idea and say that a solid moved into a fourth dimension gives us what is commonly defined as a hypersolid.

Let us use this method then in discovering the number of points, lines, planes, and three-spaces which bound the regular hypersolid that we get by moving a cube into the fourth dimension as far as the length of a side. All the sides of this figure are equal and all its angles are right angles. It is variously called a cuboid, a tesseract, or a hypercube. Beginning with a point, we may move it any chosen distance into a first dimension, noticing that the resulting figure is a line segment bounded by two points and a straight line. Moving the line into a second dimension the same chosen distance, we find that the two points trace out lines so that, with the original and final positions of the line, the resulting figure has four bounding lines. Since points are not generated by motion, the four vertices are furnished by the original and final positions of the two endpoints of the straight line. Moving this square into the third dimension the proper distance, we get a cube bounded by eight points or vertices, twelve lines or edges, and six planes or faces, since lines generate planes. In an analogous manner we get the hypercube bounded by sixteen points, thirty-two lines, twenty-four planes, and eight cubes.

The results may be put in tabular form as follows:

		Bounding Elements			
Figure	Dim.	Points	Lines	Planes	Cubes
Line	1	$2\times1=2$	1		
Square	2	$2\times2=4$	$2\times1+2=4$	1	
Cube	3	$2\times4=8$	$2\times4+4=12$	$2\times1+4=6$	1
Cuboid	4	$2\times8=16$	$2\times12+8=32$	$2\times6+12=24$	$2\times1+6=8$

The rule for obtaining the number of bounding elements in any case may be stated with perfect generality as follows: The number of bounding elements of n dimensions of any figure is derived from the bounding elements of the figure just preceding it in order by the process of multiplying its bounding elements of n dimensions by two (to get the number furnished by the original and final positions of this figure) and adding the number of bounding elements of $n-1$ dimensions of this same preceding figure (for every element of $n-1$ dimensions generates an element of n dimensions in the next figure as given in the table). For example, a hypercube is bounded by the two cubes representing the original and final positions of the generating cube and the six other cubes generated by the six planes of the original cube—a total of eight cubes in all.

Thus far, we have taken up some of the more imaginative phases of the fourth dimension and extended some of the phases of elementary high school geometry by means of it. It seems fitting that something be said about its cause for introduction and the place it is filling in the world of mathematics and science today. Although a few thinkers scattered over history as far back as Ptolmey (139 A.D.) had given some thought to the number of dimensions, the fourth dimension was not given much consideration until people began to feel the need for graphing equations of four unknowns. The fact that in mechanics one of these unknowns is often *time* probably led to Lagrange's conception of mechanics as a geometry of four dimensions and to the common idea that the fourth dimension is time. When people began to investigate the possibilities of a world in which four dimensions actually existed they discovered that a three dimensional figure could be converted into its symmetric by simply being turned over in such a world, whereas, if the figure is completely closed, no amount of skill and energy can accomplish this in three dimensional space. Since this is true, the fourth dimension has been suggested as a possible explanation for certain isomers in organic chemistry whose molecules are symmetric to each other and neither of which can be converted into the other except by chemical means.

In conclusion, let me say that this article is written chiefly for the enlightenment of high school geometry teachers and to arouse an interest that will lead to further study of this modern phase of their subject. May we each be on the alert for opportunities to use the fourth dimension to broaden and strengthen the student's conception of space, to extend the possibilities of analogy, and to make use of the tremendous imaginative interest it holds in motivating the study of geometry.

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Some Values of the Study of Mathematics*

By ALAN D. CAMPBELL, *Syracuse University*

PROBABLY YOU HAVE all had experiences like the following. We were having a week of review in a class in algebra. I asked if the students had any questions. One boy asked, "What is this all about?" Another student asked, "What is the use of all this mathematics?"

I answered the first question jokingly by quoting Bertrand Russell's remark that mathematics is the subject where you don't know what you are talking about or whether what you say is true.

The second question set me thinking about what really are some of the values of the study of mathematics. There seem to me to be five classes of these values namely the practical values, the mental values, the moral values, the spiritual values, and the aesthetic values. I shall take these up in order.

People in general judge mathematics by its practical values, but not so the mathematician himself. He studies mathematics for its own sake. His attitude is much like that of a famous mathematics professor at Johns Hopkins University named Sylvester who was lecturing on algebra. When he came to one especially theoretical topic, he said, "Thank God, this theorem can have no practical application." I don't know what theorem he was talking about, but I'll wager he was dead wrong, because mathematics is having such astonishing applications now a days. The Bell Telephone Laboratories and the General Electric Company (to mention only two of many examples) hire mathematicians to do research for them. The former are getting out wall charts showing how some really advanced and abstract topics in mathematics are being used in the solution of problems in electricity.

At one time it looked as though mathematics would decline with Greek and Latin. In fact someone asked me if I didn't have to read Euclid in Greek, and to write a Ph.D. thesis on the life of some ancient mathematician. About 1914 people were saying that

* An address given before an Onondaga County (New York) meeting of science teachers, Sept. 24, 1930.

trigonometry and advanced algebra ought to be entirely thrown out of our seconday schools, and only a bare minimum of practical arithmetic, algebra, and plane geometry taught.

Just then the World War started, and with it the teaching of mathematics took on a new lease of life. Navigation called for spherical trigonometry. Many of us had to teach spherical trigonometry without having studied the subject. Aviation and the theory of firing big guns called for mathematics through the calculus. Many of us instructors in mathematics tried to get into the infantry during the war (in order to get a rest from all mathematical studies) and found ourselves shoved into heavy or field artillery or into ordnance.

After the war there came a slump in the interest in mathematics. But now again the subject has come to the fore. It is not true (as someone remarked with sarcasm) that mathematicians are busy teaching students to become mathematicians so as to teach other students to become mathematicians, and so on forever. On the contrary, we find lawyers, doctors, chemists, zoölogists, psychologists, economists, and many others are all urging students in their fields to study mathematics through the calculus and even beyond that.

Mathematics has been found to be firmly imbedded in our lives and in the world around us. A study of the history of mathematics will bring out the close relation between mathematics and civilization itself. The invention (or discovery, whichever we wish to call it) of the Arabic numerals made possible our tremendous advance in finance and commerce. To appreciate this fact, just try to keep your accounts in the Roman numerals. The perfection of the algebraic notation was necessary for the development of physics, chemistry, engineering, and so forth. These are only two of the simplest cases showing the fundamental importance of mathematics.

The practical value of mathematics has never been better stated than in an editorial entitled "The Key of the Universe" in The Saturday Evening Post for June 1, 1929.*

Let us go on now to the mental values of the study of mathematics. The old idea of mental discipline was some years ago quoted as the best argument for the position of mathematics in the school curriculum. The text books of those days show the

* See also The Mathematics Teacher, February 1930, pp. 80–83.

earmarks of this idea, being full of difficult topics and problems that have no practical value. (In fact, it is very interesting and enlightening to compare the texts of different decades.) To the surprise of many people, when the foundation of mental discipline crumbled away, mathematics as a study still towered high like a castle in the air.

The general run of mathematicians do not know perhaps as much as they ought to about psychology, but they still believe there is something of importance in this idea of mental discipline and of the transfer of training. Of course you will find even some mathematicians who cannot keep straight their accounts or their bridge scores, and some others who are logical in geometry and very illogical in the practical affairs of every-day life. However, this anomaly can be explained by the fact that feelings and passions enter into the conduct of life and maybe into bridge and bank accounts, but never into pure mathematics.

Anyhow, "a rose by any other name will smell as sweet," and, similarly, logic is the same wherever it is found, whether in a legal argument, in a piece of mathematical reasoning, in a medical diagnosis, or hidden away in some other of life's activities. So for this reason, mathematics, because of its being pure logic (stripped of all the trimmings of sentiment, prejudice and the like) stands out as a fine subject with which to train the mind. A student who has worked an original exercise in geometry or an applied problem in algebra and has gone through the whole process of searching for a method of attack, then finding theorems or formulas that fit the case, then finishing the task in all its details—this student ought to feel a thrill not unlike that of a scientist or some other explorer after truth. Such a student has travelled along the road of creative logic and reasoning and discovery just as surely as can be.

This brings us naturally to our next topic, namely the moral value of the study of mathematics. I remember reading in a book on moral training that if a person feels his thoughts are leading him into some vice or crime, he should work out a simple problem in mathematics and he will then be rescued from the temptation. In other words (you might ask) are we to drive out the devil by mathematical curses and charms? No, not exactly that, because the above advice (though rather impractical) is nevertheless sound. Mathematical studies are free from all prejudices, passions, senti-

ments, and feelings. Democratic mathematics is the same as Republican, Methodist mathematics is the same as Catholic or Hebrew, Chinese mathematics is the same as American.

In the pursuit of mathematics we come as close to the perfectly moral action as we ever do. We must attack a mathematical problem with patience, modesty, love for truth, absolute surrender to the laws of the subject, absolute mental and moral candor and honesty. In many other activities we usually rationalize our conduct. That is, we are led to do something by a conscious or subconscious lower motive, and then (often unconsciously) we ease our conscience by inventing good reasons for the action. For example, we may dislike a man because he has hurt our pride or made us feel inferior; but we soon find other reasons for our dislike such as his real or supposed immoral conduct or too radical views. On the other hand if we liked the man we would overlook his faults.

In the study of mathematics, on the contrary, our guiding reasons and principles are all open and above-board. In some other studies our prejudices often trick us into ignoring important facts, or they even color our whole attitude toward the subject. Compare, for instance, the Protestant and Catholic accounts of the Reformation, the Northern and Southern histories of the Civil War, the British and American descriptions of the Revolution, the Capital and the Labor theories of economics. In desperation we ask "Which is the true account, or are both of them false?" Never do we run into this dilemma in mathematics.

The above-mentioned abiding truth and permanent character of mathematics constitute what I consider one of the chief spiritual values of its study. Here we have a body of knowledge that is true everywhere and forever. We never have to retrace our steps in the exploration of the subject. Euclid's theorems are as true now in English as they ever were in Greek. Mathematics is like the house builded upon the rock (according to the parable in the Bible). Many other subjects resemble the house builded upon the sand.

The fundamental ideas and rules of procedure in mathematics are the same yesterday, to-day, and forever. Theorems have been discovered in one country and century, and forgotten. Then in another country and century these theorems have been rediscovered. They turn out to be exactly the same theorems, and the same logic was used in the proof.

In these years of the shifting of moral standards, of the changing of religious ideas, of the attacks on free governments, of the very revolutions in our viewpoints on history, politics, and even science, we need somewhere to place our feet solidly. This is perhaps a slightly pharisaic attitude, but the mathematician rather glories in the fact that he is "not like one of these" but that he has just such a firm foundation under his life and his studies. You should emphasize to the students this permanence of mathematics. You may thereby be giving them the same feeling of security in a changing world that the mathematician possesses. And this feeling of security and permanence may help our young people in finding a footing on the slippery ledges that lead to character and attainment in life.

Another spiritual value of the study of mathematics lies in the cultivation thereby of a taste for intellectual pleasure in place of mere physical pleasure. We do not believe in living for pleasure alone, but the fact remains that we must find pleasure and satisfaction in what we do and in our way of life or we cannot go on. Now a days we see so many people madly pursuing physical pleasures, when we know that intellectual pleasures are keener and more lasting. Let us cultivate in our students a taste for the pleasures to be found in the study of mathematics and we shall have done them a great spiritual service. We want to see our students' faces light up with pleasure and enthusiasm over a beautiful piece of mathematical reasoning.

The words "beautiful piece of mathematical reasoning" bring me to the value that the mathematician himself ranks as the highest, namely the aesthetic value of the study of mathematics. Men must find beauty and pleasure in their surroundings and in their work, or they perish.

The story is told of an old watchmaker in New York City who was compelled by his daughter to shut up shop and travel to the Pacific coast and back. He looked at all the scenery and took in all the side-trips. Then with a sigh of relief he returned and reopened his shop. A customer came in with an old Swiss watch that had stopped running. The watchmaker opened up the watch, and his eyes brightened at once. He turned to the customer and said, "I saw some nice views out west, they all looked pretty much alike. But this is the most beautiful mainspring I ever saw."

Some day you may see a mathematician while looking at

Niagara Falls get a far-away look in his eyes. Then he will pull out a piece of paper and a pencil, jot down something, and puzzle over it a minute. Then the frown will leave his face and with a look of triumph and delight he will put the paper back into his pocket and return with more peace of mind to enjoy the scenery before him.

This serves to illustrate how much a mathematician loves his subject and how much satisfaction he gets from even a slight success in it. The students know very well how much a mathematician loves the beauty of his subject. When they do not know their lessons, they try to get him to talk about the delights of the fourth dimension, or the marvels and mystery of infinity, or the perplexities of Einstein's theory.

I could talk for hours about the beauties of mathematics. It is mathematics as an art and not as an industry that draws the mathematician. Mathematics and music and poetry are often combined in a mathematician. A little book called "The Philosophy of Mathematics" by James B. Shaw is full of eloquent tribute to mathematics. In one place, the writer describes mathematics as a growing tree whose roots dive deep into the mind of man and into the realm of nature, whose trunk rises sturdily, and whose branches spread out gloriously in every direction. This tree, he says, is growing constantly everywhere, in the roots, in the trunk, and in the branches.

The National Council of Teachers of Mathematics Twelfth Annual Meeting

Detroit, Michigan

February 20th and 21st, 1931

Detroit-Leland Hotel

General Theme:

THE RELATION OF MATHEMATICS TO MODERN THOUGHT

Friday Evening, February 20th

7:45 P.M. Eastern Standard Time

The National Council of Teachers of Mathematics, Guest and Servant of the Detroit Public Schools.

E. L. Miller, Assistant Superintendent of Detroit Public Schools, in charge of High Schools, and formerly president of The National Council of Teachers of English.

Suggestions Concerning the Future of the National Council, William Betz, Department of Mathematics, Public Schools of Rochester, New York.

The Need for being Definite with Respect to Achievement Standards,

Raleigh Schorling, Professor of Education, School of Education, University of Michigan, and formerly president of the National Council of Teachers of Mathematics.

Saturday, February 21st

9:00 A.M. Eastern Standard Time

Address.—Revision of Instruction in Geometry in Secondary Schools. Professor Ralph Beatley, Harvard University.

Note.—Meetings of the Council are open to members and their friends, without any formality, but all persons are requested to register their attendance with the clerk who will be stationed near the entrance to the room in which the meetings are held.

Reports of committees investigating the status and methods of teaching geometry, with special reference to combining plane and solid geometry,

C. M. Austin, Department of Mathematics, Oak Park High School, Oak Park, Illinois, and formerly president of the National Council of Teachers of Mathematics.

Discussion.

11:00 A.M.

Annual business meeting, including election of officers and the report of the Secretary-Treasurer.

2:00 P.M. Eastern Standard Time

Algebra, the Special Training Ground of the Reason,

Suzanne K. Langer (Mrs. W. L. Langer), Cambridge, Mass.

Mathematics as the Subject Widens the Mind's Universe,

N. H. Anning, Professor of Mathematics, University of Michigan.

Symptoms of Mathematical Powers,

W. W. Rankin, Professor of Mathematics, Duke University, Durham, N. C.

6:00 P.M. Eastern Standard Time

Banquet.

The Training of Teachers of Mathematics with special reference to the Relation of Mathematics to Modern Thought,

E. R. Hedrick, Professor of Mathematics, University of California at Los Angeles, and Chairman of the American Committee of the International Commission on the Teaching of Mathematics.

A presentation of a new talking motion picture film upon mathematics—The Play of Imagination in Geometry.

By David Eugene Smith and Aaron Bakst.

Also a talking motion picture showing how such pictures are made.

Suggestions to Contributors

It will help the reader if titles of articles submitted for publication in *The Mathematics Teacher* be as descriptive as possible A specific title is more useful than a general one. A short title is often better than a long one. Consult the index of *The Mathematics Teacher* both the cumulative one and that for the past year to see whether the title you have chosen has been used before. If so, can you think of another that fits your article more closely?

If you quote from the work of some other writer, be sure to give the author's name in full, the title of the book, its publisher and date and the page reference. If you quote from a magazine article, give the author's name, title of the article, name of the magazine, volume number, and page reference.

It will help the editor if manuscripts whether typed or written have margins at least $1\frac{1}{2}''$ wide at the left and $1''$ at the right. Typed manuscripts should be double spaced. If your work includes formulas or equations, it is helpful if these be written on a separate line. The author's name and teaching position should appear on the first page. Manuscripts should be carefully proof read, and the spelling of proper names should be checked with special care. Titles of books, names of magazines, and the letters used in algebraic or geometric formulas should be underscored so that the printer will set them up in italics. Quotation marks should be used with the titles of magazine articles. Geometric diagrams and other cuts are expensive. Accordingly, only those that are really necessary can be used.

It will improve your article if you read it carefully to eliminate expressions from current slang or those that are colloquial for while these may be effective in addressing an audience, they detract from the dignity of a written essay. Further, if your paper was originally written for presentation before a group, read it carefully to see what references to "the audience of this afternoon" should be omitted in publication. But be sure also to add a note that this paper was presented at the meeting of such and such an organization on such and such a date.

It will greatly assist the editor and the readers if these suggestions are followed, but remember that *The Mathematics Teacher* is interested in the articles that interest you.

NEWS NOTES

THE PLAY OF IMAGINATION IN GEOMETRY

An Educational Talking Picture by Professor David Eugene Smith in Collaboration with Aaron Bakst

Produced and Released by Electrical Research Products Inc., New York City

Section 19 of the New York Society for the Experimental Study of Education is the Mathematics Section of the Society. It is a vigorous section, which holds its monthly meetings in the Faculty Club house of Columbia University, where a dinner precedes the feast in Mathematics. This Section is also a local branch of the National Council of Teachers of Mathematics. As a rule, approximately one hundred attend these dinner meetings, but more than two hundred diners were present on Saturday evening October 25, 1930, and before the feature of the evening began, an additional three hundred crowded into the hall.

There was a reason for this outpouring. An educational film in every sense of the term worthy of the title, was given its premiere showing, and the originator, Professor David Eugene Smith, spoke after the showing. In the crowded dining hall, with many standing throughout the exercises, there was rapt attention as the flashing play of forms and relations swept across the screen.

The permanence of mathematical relations or laws, as the geometric elements changed size and position was made evident as the continuous change was shown on the screen. So simple a problem as tracing the point of intersection of two lines as they were pivoted about points in the observer's region of the plane, so that the angle at the point of intersection became smaller, and the point receded to infinity, became a living thing. The voice of Professor Smith, clear almost as his own voice, made clarifying comment on each problem and mutation.

All the studies presented were dynamic. The geometry of Euclid was static, and because of the great authority of the classical tradition, geometry has continued to be a static subject down to the present time. And this is true in spite of the fact that the spirit of modern peoples is the very antithesis of that of the ancient Greeks and Romans. They were interested in the *thing become*, as the philosophers would say, while the moderns are interested in the *thing becoming*. Witness the fact that modern mathematics, from calculus on, treats dominantly of rates of *change* and dynamical problems. Functions and functional investigations deal continually with related changes. The Greeks (Euclid) lived happily in the *finite* and dreaded the infinite. The nineteenth century saw the development of the mathematics of *infinity*, and the solution of many philosophical puzzles involving the relation of part to whole in infinity. Yet most, of our secondary school students are cut off from this modern spirit because teachers

still teach geometry as a static subject, and never introduce their students to the infinite.

This introduction to the infinite in geometry was made easy and vivid by the film. Two circles with their common external and internal tangents were shown, first with the circles unequal in size. Then the smaller circle gradually grew in radius, and the points of intersection of the pairs of tangents moved across the screen in consequence. Of course the point of intersection of the external tangents rapidly disappeared as the diameter of the smaller approached the diameter of the larger, flashed to infinity on the left, jumped to infinity on the right, and came back to the finite region of the plane, as the smaller circle passed through equality and became the larger.

Then the circles with their system of common tangents approached each other, merged and emerged into changed relative positions. The common internal tangents shut together like a pair of shears, and emerged again in the changed position. Now the circles with their common tangents were rapidly rotated before our eyes, and the shadow system of a planet was born, and the imagination necessary to understand an eclipse was kindled in the beholder with less resistance on the part of "our dull clay" than any other teaching device would encounter. Part of the cosmic problem that confounded the ancients was laid bare so that even a child could comprehend.

Words are inadequate to convey a visual impression, or portray its effects. But we saw a triangle come to life, and two of its sides rotate so that their intersection point described a circle; we saw the sides c and a of a triangle move so that the angle C changed from acute, to right, to obtuse, to straight, and saw the relation $c^2=a^2+b^2-2bp_a$ remain true in the midst of these changes, since p_a became zero and then negative. It was a revelation of the permanence of mathematical relations and forms in the midst of striking changes. The thrill and the charm of the film to a lover of mathematics can not be realized fully until the picture is seen.

W. S. Schlauch

The thirty-ninth regular meeting of the Association of Mathematics Teachers of New Jersey was held at the State Teachers College at Upper Montclair, N. J. on Oct. 18, 1930.

The following program was rendered:

Morning Session

"The Inception of Geometric Knowledge," Mr. Harrison E. Webb, Market Street High School.

"Unit Difficulties in Plane Geometry," Mr. Virgil S. Mallory, Montclair State Teachers College.

Discussion led by Professor John C. Stone, Montclair State Teachers College.

"Ability Grouping in Ninth Grade Algebra," Mr. Ferdinand Kertes, Perth Amboy High School.

Afternoon Session

"An Objective Test in Plane Geometry," Mr. William W. Strader, Dickinson High School, Jersey City.

"The Use of Instruments as an Aid in the Teaching of Plane Geometry," Mr. Carl N. Shuster, State Teachers College, Trenton.

"The Use of Motion Pictures and Models in Mathematics Class Rooms," Mr. Aaron Bakst.

Discussion led by Mr. Howard F. Hart, East Orange High School.

The fall meeting of the Connecticut Valley section of the Association of Teachers of Mathematics in New England was held at the High School in

Westfield, Massachusetts on Saturday, Nov. 8, 1930. The following program was rendered:

Morning Session

10:30 Informal Reception.

10:45 Do's and Don't's in Writing College Entrance Examinations in Geometry.
Fred D. Aldrich, Senior Master and Head of Department of Mathematics, Worcester Academy.
The Content of Geometry.
Professor W. R. Longley, Yale University.

12:30 Luncheon at the School Cafeteria.

Afternoon Session

1:30 Business Meeting.

1:45 From the Side Lines.
Harry B. Marsh, Principal of Technical High School, Springfield.
Some Modern Tendencies in Teaching, Criticism and Comment.
George W. Creelman, Hotchkiss School, Lakeville.

The Mathematics Section of the New York State Teachers Association was held at Teachers College in New York on Friday afternoon Oct. 31, 1930.

The following program was given:

"Teaching Algebra as a Method of Thinking," Professor W. D. Reeve, Teachers College.

General Discussion led by Mr. F. Eugene Seymour, Supervisor of Mathematics State Education Department.

The Mathematics Section of the Minnesota Education Association held its annual meeting in St. Paul on Oct. 31, 1930.

The following program was given:

"An Experimental Evaluation of the Individualized Teaching of Algebra," C. N. Stokes, University High School.

"Teachers' Reports of Supervisory Needs in the Teaching of Arithmetic," W. E. Peik, University of Minnesota.

"Problems in Making the Curriculum in Mathematics," Ernest Horn, University of Iowa.

In one of the courses in Statistics at Teachers College, Columbia University recently 186 students reported background experience with mathematics as follows:

	Number	*Percent*
Algebra (ninth)......	186	100
Plane geometry......	185	99.5
Eleventh grade algebra	140	75.3
Trigonometry.......	128	69.6
College Algebra......	112	60.2
Analytical geometry..	60	32.3
Differential calculus..	42	22.6
Integral Calculus....	36	19.4
Theory of Probability.	8	4.3
Other mathematics courses...........	23	17.7

This should be of interest to those who feel that the mathematics preparation of many of our advanced students is deficient.

At Drake University, Des Moines, Iowa on November 14, 1930 the following mathematics program was given:

Senior High School Mathematics, Mr. Orville A. George, Mason City, Iowa.

A Proposed Arrangement of High School Courses in Geometry, Professor Dunham Jackson, University of Minnesota.

The Fall meeting of the Association of Teachers of Mathematics in New England was held at Boston University on December 6, 1930 with Mr. Raymond Morley presiding. The following program was rendered:

10:30 A.M.

Standardized Tests: Why or Why Not? Miss Lena G. Perrigo, Memorial High School for Girls, Roxbury.

The Effect of Road Curvature and Speed on Safety of Automobile Operation, Prof. Francis W. Roys, Worcester Polytechnic Institute.

2 P.M.

A Mathematical Theory of Harmony and Melody, Prof. George D. Birkhoff, Harvard University. Assisted at the piano by Prof. Marston Morse, Harvard University.

Conservation of Energy in High School Mathematics, Prof. Ralph Beatley, Harvard University.

NEW BOOKS

Changes in the Content of Elementary Algebra since the Beginning of the High School Movement as Revealed by the Textbooks of Period. By Amy Olive Châteauneuf. Privately printed, Philadelphia, 1929. X+191 pp.

This work attempts to answer the question: "What was taught in elementary algebra in the past, not just in the immediate past, but approximately since algebra has been a high school subject?" It attempts to answer this question in so far as an answer is contained in the contemporary algebra textbooks of the various decades since the inception of the high school movement. A total of 257 books by 158 authors constitutes the basis of the study. These are listed in a bibliography in the appendix and show the wide range of the study.

The book is a veritable mine of information on the various topics which it treats and should prove invaluable to all teachers interested in the development of algebra in American secondary schools over a period of more than one hundred years from 1818 to 1928. It covers the following broad general topics: Exercises; Equations; Roots and Radicals; Fractions; General Equations and Formulas; Multiplication; Addition and Subtraction; Division; Proportions and Progressions; Negative Exponents; Parentheses; Factoring; Factors and Multiples; and Graphs.

Every topic is treated with the utmost thoroughness. The plan used in presenting each one may be shown by a single illustration. In the first part of the chapter on equations are given the tables and graphs illustrating the development by decades throughout the period examined. The first table gives the mean number of exercises in equations per book, together with a comparison of the increase or decrease in each decade. This is followed by a table showing the percentage of exercises in equations, together with a comparison of the increase or decrease in each decade. A graph accompanies this table. The next table gives the distribution of exercises in equations according to the fifth in which they appear in the respective books. Then come tables on

percentage of simple and complicated exercises in equations for each decade with a graph and on percentage of exercises devoted to linear, quadratic, simultaneous, radical and unclassified equations, respectively, with a graph of striking interest. A discussion of these tables points out the outstanding features of them and a valuable summary closes the topic.

The work contains in all 78 tables and 39 graphs. One may turn to the final summary of the whole matter to find, for example, the periods of greatest complexity and simplicity for the topics of elementary algebra and also the decade in which started the present movement toward simplicity. One turns with interest to the section on factoring to find the answer to the until-now-unanswered question: "Just when did factoring spring into prominence in textbooks?" The graph shows that the peak was reached in the decade beginning 1890 and in the one from 1920 on, the graph runs down to the same height it had attained in the 1880 decade. Factoring had a short life of 40 years but its life was all too long at that.

This contribution to the history of education was prepared as a doctor's thesis for the University of Pennsylvania. It will, therefore, be found in every college and public library throughout the country. It should be widely consulted because of its intrinsic interest and also because of the light which it throws on present day educational problems.

Lao G. Simons

Hunter College
New York

OFFICIAL BALLOT

For the Election of Officers at the February 20, 1931, Meeting of the National Council of Teachers of Mathematics

For Second Vice-President, 1931-1933 (Vote for One)

☐ Hildebrandt, Martha
Maywood, Illinois

☐ Taylor, E. H.
Charleston, Illinois

For Members of the Board of Directors, 1931-1934 (Vote for Three)

☐ Gugle, Marie
Columbus, Ohio

☐ Sabin, Mary S.
Denver, Colorado

☐ Orleans, Joseph B.
New York, New York

☐ Thiele, C. Louis
Detroit, Michigan

☐ Poole, Hallie S.
Buffalo, New York

☐ Weimar, M. Bird
Wichita, Kansas

Please mark this ballot at once and mail same to Edwin W. Schreiber, Secretary, 314 N. Ward St., Macomb, Illinois.